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On Basicity of the Perturbed System of Exponents in Morrey-Lebesgue Space

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Abstract. In this work, the double system of exponents with complex-valued coefficients is considered. Special cases of these systems often arise when solving problems of mathematical physics by Fourier method. A lot of research by Paley-Wiener, N. Levinson and others have been dedicated to the basis properties of these systems. We find conditions on the coefficients of the system which guarantee its basicity in Morrey-Lebesgue type spaces.

Key Words and Phrases: Morrey-Hardy classes, Riemann problem, bases, double system of exponents.

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1. Introduction

The Morrey spaces were introduced by C. Morrey in 1938. Since then, various problems related to these spaces have been intensively studied. Playing an important role in the qualitative theory of elliptic differential equations (see, for example, [1, 2]), these spaces also provide a large class of examples of mild solutions to the Navier-Stokes system [3]. In the context of fluid dynamics, Morrey spaces have been used to model fluid flow when vorticity is a singular measure supported on some sets in R^n [4]. There appeared lately a large number of research works which considered fundamental problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc in these spaces (see, for example, [5] and the references above). More details about Morrey spaces can be found in [6, 7].

In view of the aforesaid, there has recently been a growing interest in the study of various problems in Morrey-type spaces. For example, some problems

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of harmonic analysis and approximation theory have been considered in [8, 9, 10, 11, 12].

Note that when solving many problems of mathematical physics by the Fourier method [13, 14, 15, 16], there often arise perturbed systems of sines and cosines of the following form:

$$\{\sin(nt + \alpha(t))\}_{n\in N}, \tag{1}$$

$$\{\cos(nt + \alpha(t))\}_{n\in N}, \tag{2}$$

where $\alpha(t) = \frac{1}{2}(\beta\, t + \gamma)$, $\beta,\ \gamma \in R$ are real parameters, and N is the set of all natural numbers. The justification of this method requires the study of basis properties (completeness, minimality, basicity of the above systems in Lebesgue and Sobolev function spaces. Their basis properties in Lebesgue spaces $L_p(-\pi, \pi)$ with $\gamma = 0$, have been comprehensively studied in [17, 18, 19, 20, 21]. The weighted case of L_p has been considered by E.I.Moiseev [22, 23]. Basis properties of some perturbed systems of exponents in Sobolev spaces have been studied in [27, 28, 29, 30]. See also [31, 32, 33, 34, 35, 36].

One of the methods to study basis properties of systems like (1), (2) is a method of boundary value problems of the theory of analytic functions. It dates back to a note by A.V. Bitsadze [37]. This method was successfully used in [18, 19, 20, 21, 22, 23, 24, 25, 26]. To apply this method to the study of basis properties of systems like (1), (2) in Morrey-type spaces, we first have to investigate the solvability of the Riemann boundary value problems in Morrey-type Hardy spaces.

In [11], the Morrey-Hardy and Morrey-Lebesgue classes have been treated and their subspaces have been defined with the shift operator being continuous. Basicity of the classical system of exponents and some of its parts in these subspaces has been studied.

Moreover, the homogeneous and non-homogeneous Riemann boundary value problems have been considered in Morrey-Hardy classes in [38]. Conditions on the coefficient of the problem are obtained which are sufficient for the Noetherness of the considered problems in these classes.

In this work, we consider the Riemann boundary value problem in Morrey-type Hardy spaces. We study the solvability of this problem and construct a general solution for both homogeneous and non-homogeneous problems under some conditions on the coefficients of the problem. The obtained results are applied to the study of basicity properties of double system of exponents with complex coefficients. Using the results of [38], we derive a sufficients condition for the basicity of this system in Morrey-Lebesgue type spaces.

2. Needful Information

We need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_{\Gamma}$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$.

The expression $f(x) \sim g(x)$, $x \in M$, means

$$\exists \delta > 0 : \delta \leq \left| \frac{f(x)}{g(x)} \right| \leq \delta^{-1}, \forall x \in M.$$

A similar meaning will be attached to the expression $f(x) \sim g(x)$, $x \to a$.

By the Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0 \leq \alpha \leq 1$, $p \geq 1$, we mean a normed space of all functions $f(\cdot)$ measurable on Γ, equipped with a finite norm $\|\cdot\|_{L^{p,\alpha}(\Gamma)}$:

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_{B} \left(\left| B \bigcap \Gamma \right|_{\Gamma}^{\alpha-1} \int_{B \bigcap \Gamma} |f(\xi)|^{p} |d\xi| \right)^{1/p} < +\infty.$$

$L^{p,\alpha}(\Gamma)$ is a Banach space and $L^{p,1}(\Gamma) = L_{p}(\Gamma)$, $L^{p,0}(\Gamma) = L_{\infty}(\Gamma)$. Weighted version of the Morrey-Lebesgue space $L_{\mu}^{p,\alpha}(\Gamma)$ on Γ with a weight function $\mu(\cdot)$ and a norm $\|\cdot\|_{L_{\mu}^{p,\alpha}(\Gamma)}$ can be defined in a natural way:

$$\|f\|_{L_{\mu}^{p,\alpha}(\Gamma)} = \|f\mu\|_{L^{p,\alpha}(\Gamma)}, f \in L_{\mu}^{p,\alpha}(\Gamma).$$

The embedding $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_{1}(\Gamma)$, $\forall \alpha \in [0,1]$, $\forall p \geq 1$. The case of $\Gamma \equiv [-\pi, \pi]$ will be denoted by $L^{p,\alpha}(-\pi, \pi) \equiv L^{p,\alpha}$.

By S_{Γ} we denote the following singular integral operator:

$$(S_{\Gamma} f)(\tau) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)\, d\zeta}{\zeta - \tau}, \ \tau \in \Gamma.$$

Unit circle centered at $z = 0$ will be denoted by γ with $int\, \gamma = \omega$. Define the Morrey-Hardy space $H_{+}^{p,\alpha}$ of functions $f(z)$ analytic inside ω with a norm $\|\cdot\|_{H_{+}^{p,\alpha}}$:

$$\|f\|_{H_{+}^{p,\alpha}} = \sup_{0<r<1} \left\| f\left(re^{it}\right) \right\|_{L^{p,\alpha}}.$$

The following theorem was proved in [11].

Theorem 1. *The function $f(\cdot)$ belongs to $H_{+}^{p,\alpha}$, $1 < p < +\infty$, if and only if $f^{+} \in L^{p,\alpha}$ and the following Cauchy integral formula is true:*

$$f(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f^{+}(\tau)\,d\tau}{\tau - z},$$

where $f^{+}(\cdot)$ *are nontangential boundary values of* $f(\cdot)$ *on* γ.

The analog of Smirnov's theorem in Morrey-Hardy classes is also true:

Theorem 2. *Let* $f \in H_{+}^{p_1,\alpha}$, $1 \le p_1 < +\infty$, $0 \le \alpha \le 1$, *and* $f^{+} \in L^{p_2,\alpha}$, *where* $p_1 < p_2 < +\infty$, f^{+} *are nontangential boundary values of the function* f *on* γ. *Then* $f \in H_{+}^{p_2,\alpha}$.

Denote by $\tilde{L}^{p,\alpha}$ the linear subspace of $L^{p,\alpha}$ consisting of functions whose shifts are continuous in $L^{p,\alpha}$, i.e. $\|f(\cdot+\delta) - f(\cdot)\|_{L^{p,\alpha}} \to 0$ as $\delta \to 0$. The closure of $\tilde{L}^{p,\alpha}$ in $L^{p,\alpha}$ will be denoted by $\mathcal{M}^{p,\alpha}$. The following theorem was proved in [11]:

Theorem 3. *Infinitely differentiable functions on* $[0, 2\pi]$ *are dense in the space* $\mathcal{M}^{p,\alpha}$.

Similarly, define the $\mathcal{M}_{\rho}^{p,\alpha}$ space with the weight $\rho(\cdot)$. Denote by $\tilde{\mathcal{M}}_{\rho}^{p,\alpha}$ the set of functions whose shifts are continuous in $L_{\rho}^{p,\alpha}$, i.e.

$$\|f(\cdot+\delta) - f(\cdot)\|_{p,\alpha;\rho} \to 0, \quad \delta \to 0.$$

$\tilde{\mathcal{M}}_{\rho}^{p,\alpha}$ is a linear subspace of $L_{\rho}^{p,\alpha}$. The closure of $\tilde{\mathcal{M}}_{\rho}^{p,\alpha}$ in $L_{\rho}^{p,\alpha}$ will be denoted by $\mathcal{M}_{\rho}^{p,\alpha}$. It is easy to see that if $\rho \in L^{p,\alpha}$, then $C[-\pi,\pi] \subset \mathcal{M}_{\rho}^{p,\alpha}$.

The following lemma plays an important role in obtaining our main results.

Lemma 1. *Let* $f(\cdot) \in L_{\infty}$; $g(\cdot) \in \mathcal{M}^{p,\alpha} \wedge 1 \le p < +\infty$, $0 < \alpha \le 1$. *Then the inclusion* $f(\cdot)\,g(\cdot) \in \mathcal{M}^{p,\alpha}$ *is valid.*

Consider the following singular operator

$$(Sf)(\tau) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)\,d\xi}{\xi - \tau}, \tau \in \gamma.$$

Using the results of [9, 10, 12], it is easy to prove the following

Theorem 4. *Singular operator* S *acts boundedly in* $\mathcal{M}^{p,\alpha}(\gamma)$ *when* $0 < \alpha \le 1$ *and* $1 < p < +\infty$.

The following theorem can also be proved.

Theorem 5. *Let $f \in \mathcal{M}^{p,\alpha}$, $0 < \alpha \leq 1$, $1 < p < +\infty$. Then*

$$\left\| (\mathcal{K}f)(r\xi) - f^{+}(\xi) \right\|_{L^{p,\alpha}} \to 0, r \to 1-0,$$

where $(\mathcal{K}f)(z)$ is a Cauchy type integral

$$(\mathcal{K}f)(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)\, d\xi}{\xi - z}, z \notin \gamma.$$

The similar assertion is also true for $f^{-}(\xi)$ as $r \to 1+0$, where $f^{-}(\cdot)$ are nontangential boundary values of $f(\cdot)$ outside ω.

The final result of [11] is the following

Theorem 6. *System of exponents $\left\{e^{i\,nt}\right\}_{n\in Z}$ forms a basis for $\mathcal{M}^{p,\alpha}$ when $1 < p < +\infty$, $0 < \alpha \leq 1$.*

Consider the space $H_{+}^{p,\alpha}$. Denote by $L_{+}^{p,\alpha}$ the subspace of $L^{p,\alpha}$, generated by the restrictions of the functions from $H_{+}^{p,\alpha}$ to γ. It follows directly from the above results that the spaces $H_{+}^{p,\alpha}$ and $L_{+}^{p,\alpha}$ are isomorphic and $f^{+}(\tau) = (Jf)(z)$, where $f \in H_{+}^{p,\alpha}$, f^{+} are nontangential boundary values of f on γ, and J performs a corresponding isomorphism. Let $\mathcal{M}_{+}^{p,\alpha} = \mathcal{M}^{p,\alpha} \bigcap L_{+}^{p,\alpha}$. It is clear that $\mathcal{M}_{+}^{p,\alpha}$ is a subspace of $\mathcal{M}^{p,\alpha}$ with regard to the norm $\|\cdot\|_{L^{p,\alpha}}$. Let $\mathcal{M}H_{+}^{p,\alpha} = J^{-1}\left(\mathcal{M}_{+}^{p,\alpha}\right)$. This is a subspace of $H_{+}^{p,\alpha}$. Let $f \in H_{+}^{p,\alpha}$ and f^{+} be its boundary values. It is absolutely clear that the norm $\|f\|_{H_{+}^{p,\alpha}}$ can be also defined as $\|f\|_{H_{+}^{p,\alpha}} = \|f^{+}\|_{L^{p,\alpha}}$.

Similar to the classical case, we define the Morrey-Hardy class outside ω. So let $D = C \backslash \omega$. We will say that the function f analytic in D^{-} has finite order k at infinity, if its Laurent series in a neighborhood of the point at infinity has the following form:

$$f(z) = \sum_{n=-\infty}^{k} a_n z^n, k < +\infty, a_k \neq 0. \tag{3}$$

Thus, when $k > 0$, the function $f(z)$ has a pole of order k; when $k = 0$, it is bounded; and when $k < 0$, it has a zero of order $(-k)$. Let $f(z) = f_0(z) + f_1(z)$, where $f_0(z)$ is the main, and $f_1(z)$ is the regular part of expansion (1) for the function $f(z)$. Consequently, if $k \leq 0$, then $f_0(z) \equiv 0$. When $k > 0$, $f_0(z)$ is a polynomial of degree k. We will say that the function $f(z)$ belongs to the class ${}_{m}H_{-}^{p,\alpha}$, if f has an order at infinity less than or equal to m, i.e. $k \leq m$ and $f_1\left(\frac{1}{z}\right) \in H_{+}^{p,\alpha}$.

Absolutely similar to the case of $\mathcal{M}H_{+}^{p,\alpha}$, we define the class ${}_{m}\mathcal{M}H_{-}^{p,\alpha}$. In other words, ${}_{m}\mathcal{M}H_{-}^{p,\alpha}$ is a subspace of functions from ${}_{m}H_{-}^{p,\alpha}$, whose shifts on a unit circle are continuous with regard to the norm $\|\cdot\|_{L^{p,\alpha}(\gamma)}$.

To study the basicity of the system of exponents, we will need the following result of [11]:

Theorem 7. *Systems* $\left\{e^{i\,nt}\right\}_{n\in Z_+}$*;* $\left\{e^{-i\,nt}\right\}_{n\in N}$ *(*$\{z^{\,n}\}_{n\in Z_+}$*;*$\{z^{\,-n}\}_{n\in N}$*) form bases for spaces* $\mathcal{M}_+^{p,\alpha}$*;* ${}_{-1}\mathcal{M}_-^{p,\alpha}$ *(*$\mathcal{M}H_+^{p,\alpha}$*;* ${}_{-1}\mathcal{M}H_-^{p,\alpha}$*), respectively.*

We will use also the following concepts. Let $\Gamma\subset C$ be some bounded rectifiable curve, and $t=t\,(\sigma)\,,\ 0\le\sigma\le l$, be its parametric representation with respect to the length of arc σ, where l is the length of Γ. Let $d\mu\,(t)=d\sigma$, i.e. $\mu\,(\,\cdot\,)$ is a linear measure on Γ. Let

$$\Gamma_t\,(r)=\{\tau\in\Gamma:|\tau-t|<r\}\,,\Gamma_{t(s)}\,(r)=\{\tau\,(\sigma)\in\Gamma:|\sigma-s|<r\}\,.$$

It is absolutely clear that $\Gamma_{t(s)}\,(r)\subset\Gamma_t\,(r)$.

Definition 1. *Curve* Γ *is called a Carleson curve if* $\exists\,c>0$*:*

$$\sup_{t\in\Gamma}\mu\,(\Gamma_t\,(r))\le cr,\ \forall r>0.$$

Curve Γ is said to satisfy the chord-arc condition at the point $t_0=t\,(s_0)\in\Gamma$, if there exists a constant $m>0$, independent of t, such that $|s-s_0|\le m\,|t\,(s)-t\,(s_0)|$, $\forall t\,(s)\in\Gamma$. Γ satisfies the chord-arc condition uniformly on Γ, if $\exists\,m>0:|s-\sigma|\le m\,|t\,(s)-t\,(\sigma)|\,,\,\forall t\,(s)\,,t\,(\sigma)\in\Gamma$.

Let's state the following lemma from [12], which is interesting in itself:

Lemma 2. *[12] Let* Γ *be a bounded rectifiable curve. If the exponential function* $|t-t_0|^{\gamma}\,,\ t_0\in\Gamma$*, belongs to the space* $L^{p,\alpha}\,(\Gamma)\,,\ 1\le p<\infty,\ 0<\alpha<1$*, then* $\gamma\ge-\frac{\alpha}{p}$*. If* Γ *is a Carleson curve, then this condition is also sufficient.*

We will essentially use the following theorem of N. Samko [12]:

Theorem 8. *[12] Let the curve* Γ *satisfy the chord-arc condition and the weight* $\rho\,(\,\cdot\,)$ *be defined as follows:*

$$\rho\,(t)=\prod_{k=1}^{m}|t-t_k|^{\alpha_k}\,;\ \{t_k\}_1^m\subset\Gamma,\ t_i\ne t_j,i\ne j. \tag{4}$$

Singular operator S_Γ *is bounded in the weighted space* $L_\rho^{p,\alpha}\,(\Gamma)\,,\ 1<p<+\infty$*,* $0\le\alpha<1$*, if the following inequalities are valid:*

$$-\frac{\alpha}{p}<\alpha_k<-\frac{\alpha}{p}+1,\ k=\overline{1,m}. \tag{5}$$

Moreover, if Γ *is smooth in some neighbourhoods of the points* t_k*,* $k = \overline{1,m}$*, then the validity of the inequalities* (5) *is necessary for the boundedness of* S_Γ *in* $L_\rho^{p,\alpha}(\Gamma)$*.*

In what follows, as Γ we will consider a unit circle $\gamma = \partial\omega$. Consider the weighted space $L_\rho^{p,\alpha}(\gamma) =: L_\rho^{p,\alpha}$ with the weight $\rho(\cdot)$. Let the weight $\rho(\cdot)$ satisfy the condition (5). Then, by Theorem 8, the operator S is bounded in $L_\rho^{p,\alpha}$, i.e. $\exists c > 0$:

$$\|Sf\|_{L_\rho^{p,\alpha}} \leq C \|f\|_{L_\rho^{p,\alpha}}, \forall f \in L_\rho^{p,\alpha}.$$

Let's show that $\mathcal{M}_\rho^{p,\alpha}$ is an invariant subspace with respect to the singular operator S, if the inequalities (5) are fulfilled. It is absolutely clear that to do so it suffices to prove the continuity of the shift of S. Take $\forall\delta \in R$ and consider

$$(Sf)\left(\tau e^{i\delta}\right) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi - \tau e^{i\delta}}.$$

We have

$$(Sf)\left(e^{i\delta}\tau\right) = \frac{1}{2\pi i}\int_\gamma \frac{f\left(e^{-i\delta}\xi e^{i\delta}\right) d\left(e^{-i\delta}\xi\right)}{\xi e^{i\delta} - \tau} =$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f\left(\xi e^{i\delta}\right) d(\xi)}{\xi - \tau}.$$

It follows

$$(Sf)\left(e^{i\delta}\tau\right) - (Sf)(\tau) = \frac{1}{2\pi i}\int_\gamma \frac{f\left(\xi e^{i\delta}\right) - f(\xi)}{\xi - \tau} d\xi =$$

$$= \left(S\left(f\left(\cdot\, e^{i\delta}\right) - f(\cdot)\right)\right)(\tau).$$

Let $f \in \mathcal{M}_\rho^{p,\alpha}$. Then Theorem 8 of [12] immediately implies

$$\left\|(Sf)\left(\tau e^{i\delta}\right) - (Sf)(\tau)\right\|_{L_\rho^{p,\alpha}} = \left\|\left(S\left(f\left(\cdot\, e^{i\delta}\right) - f(\cdot)\right)\right)(\tau)\right\|_{L_\rho^{p,\alpha}} \leq$$

$$\leq C\left\|f\left(\cdot\, e^{i\delta}\right) - f(\cdot)\right\|_{L_\rho^{p,\alpha}} \to 0,\ \delta \to 0.$$

Thus, the following theorem is valid.

Theorem 9. *Let the weight function $\rho(\cdot)$ be defined by (4) with $\Gamma \equiv \gamma$. If the inequalities (5) are fulfilled, then the singular operator S acts boundedly in $\mathcal{M}_\rho^{p,\alpha}$.*

Let I be some interval and $f \in L^{p,\alpha}(I)$, $g \in L^{q,\alpha}(I)$; throughout this paper $\frac{1}{p}+\frac{1}{q}=1$. We have

$$\int_I |fg|\,dt \leq |I|^{1-\alpha} \sup_{x\in I, r>0} r^{\alpha-1} \int_{I_r(x)} |fg|\,dt = |I|^{1-\alpha} \|fg\|_{1,\alpha},$$

where $|I|$ is a Lebesgue measure of I, $I_r(x) \equiv I \bigcap (x-r, x+r)$. Applying Hölder's inequality, we obtain

$$\int_I |fg|\,dt \leq |I|^{1-\alpha} \sup_{x\in I, r>0} \left(r^{\alpha-1} \int_{I_r(x)} |f|^p\,dt \right)^{\frac{1}{p}} \times$$

$$\times \left(r^{\alpha-1} \int_{I_r(x)} |g|^q\,dt \right)^{\frac{1}{q}} \leq |I|^{1-\alpha} \sup_{x\in I, r>0} \left(r^{\alpha-1} \int_{I_r(x)} |f|^p\,dt \right)^{\frac{1}{p}} \times$$

$$\times \sup_{x\in I, r>0} \left(r^{\alpha-1} \int_{I_r(x)} |g|^q\,dt \right)^{\frac{1}{q}} = |I|^{1-\alpha} \|f\|_{p,\alpha} \|g\|_{q,\alpha}.$$

Thus, the following lemma is valid:

Lemma 3. *Let $f \in L^{p,\alpha}(I) \wedge g \in L^{q,\alpha}(I), \frac{1}{p}+\frac{1}{q}=1$, $p \in [1,+\infty)$. Then the following Hölder inequality holds*

$$\|fg\|_{L_1} \leq |I|^{1-\alpha} \|fg\|_{1,\alpha} \leq |I|^{1-\alpha} \|f\|_{p,\alpha} \|g\|_{q,\alpha}.$$

In the sequel, we will often use the following obvious lemma:

Lemma 4. *Let $|f(t)| \leq |g(t)|$ for almost every $t \in [-\pi,\pi]$. Then $\|f\|_{L_\rho^{p,\alpha}} \leq \|g\|_{L_\rho^{p,\alpha}}$.*

To obtain our main result, we will also use the following lemma that follows directly from Lemma 2 of [12]:

Lemma 5. *Let $\{t_k\}_1^m \subset [-\pi,\pi]$. The finite product $\omega(t) = \prod_{k=1}^m |t-t_k|^{\alpha_k}$ belongs to the space $L^{p,\alpha}$, if the inequalities $\alpha_k \geq -\frac{\alpha}{p}$, $\forall k$, are valid, where $0<\alpha<1$, $1<p<+\infty$.*

3. Homogeneous Riemann problem in Morrey-Hardy classes

We need some results from [38] concerning the solvability of the homogeneous and nonhomogeneous Riemann problems in Morrey-Hardy classes.

Let's consider the following homogeneous Riemann problem in classes $\left(H_{+}^{p,\alpha};\ {}_{m}H_{-}^{p,\alpha}\right)$:

$$\begin{cases} F^{+}(\tau)-G(\tau)F^{-}(\tau)=0\ ,\ \tau\in\gamma, \\ \\ F^{+}(z)\in H_{+}^{p,\alpha}\ ;\ F^{-}(z)\in\ {}_{m}H_{-}^{p,\alpha}\ , \end{cases} \tag{6}$$

where

$$G\left(e^{it}\right)=\left|G\left(e^{it}\right)\right|e^{i\theta(t)},\ \ \theta(t)=\arg G\left(e^{it}\right),t\in[-\pi,\pi)\,.$$

Introduce the following functions $X_{i}^{\pm}(z)$ analytic inside (with the sign +) and outside (with the sign -) the unit circle:

$$X_{1}(z)\equiv\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|G\left(e^{it}\right)\right|\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_{2}(z)\equiv\exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi}\theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\}.$$

Define

$$Z_{i}(z)\equiv\begin{cases} X_{i}(z)\,,\ |z|<1\,, \\ \left[X_{i}(z)\right]^{-1}\,,\ |z|>1\,. \end{cases}$$

Denoting $Z(z)\equiv Z_{1}(z)\,Z_{2}(z)$, we have

$$Z^{+}(\tau)-G(\tau)Z^{-}(\tau)=0,\tau\in\gamma. \tag{7}$$

Following classics, we call the function $Z(z)$ a canonical solution of the problem (6)*.*

Theorem 10. *Let the coefficient $G(\cdot)$ of the problem* (6) *satisfy the following conditions:*

i) $G^{\pm1}\in L_{\infty}(\gamma)$;

ii) $\theta(t)\equiv\arg G\left(e^{it}\right)$ is piecewise continuous in $[-\pi,\pi]$, $\{s_{k}\}_{1}^{r}:-\pi<s_{1}<...<s_{r}<\pi$ are the points of discontinuity, $h_{k}=\theta(s_{k}+0)-\theta(s_{k}-0)$, $k=\overline{1,r}$, are the corresponding jumps, $h_{0}=\theta(-\pi)-\theta(\pi)$.

If the inequalities

$$-\frac{\alpha}{q} \leq \frac{h_k}{2\pi} \leq \frac{\alpha}{p}, k = \overline{0, r}, \tag{8}$$

are fulfilled, where $p \in (1, +\infty)$ *is some number, then the homogeneous Riemann problem* (4) *has a general solution in classes* $H_+^{p,\alpha} \times {}_mH_-^{p,\alpha}$ *of the form* $F(z) \equiv Z(z) P_m(z)$*, where* $Z(\cdot)$ *is a canonical solution, and* $P_m(\cdot)$ *is an arbitrary polynomial of degree* $k \leq m$.

This theorem has the following corollary.

Corollary 1. *Let all the conditions of Theorem 10 be fulfilled. Then the homogeneous Riemann problem* (6) *has only a trivial solution in classes* $H_+^{p,\alpha} \times {}_mH_-^{p,\alpha}$*, when* $m \leq -1$.

Note that in case where the conditions i), ii) are satisfied with respect to the coefficient $G(\cdot)$, the solution of the homogeneous problem (6) belongs to the class $\mathcal{M}H_+^{p,\alpha} \times {}_m\mathcal{M}H_-^{p,\alpha}$. In fact, it follows from the expression of solution that it suffices to show that the boundary values of $Z^{\pm}(\cdot)$ belong to the space $\mathcal{M}^{p,\alpha}$. We have $Z^{\pm}(\cdot) = Z_1^{\pm}(\cdot) \times Z_2^{\pm}(\cdot)$. As $Z_1^{\pm} \in L_\infty$, it follows from Lemma 1 that it suffices to prove the validity of inclusion $Z_2^{\pm} \in \mathcal{M}^{p,\alpha}$. Lemma 1 directly implies the validity of inclusion $L_\infty \subset \mathcal{M}^{p,\alpha}$. As $\theta(\cdot) \in L_\infty$, applying Stokhotski-Plemelj formulas to $Z_2(z)$, we obtain from Theorem 8 that the inclusion $Z_2^{\pm} \in \bar{L}^{p,\alpha}$ is valid. So the following statement is true.

Statement 1. *Let all the conditions of Theorem 7 be satisfied. Then the solution of the problem* (6) *belongs to the class* $\mathcal{M}H_+^{p,\alpha} \times {}_m\mathcal{M}H_-^{p,\alpha}$.

Remark 1. *It should be noted that for* $\alpha \to 1-0$ *the inequalities* (8) *become*

$$-\frac{1}{q} < \frac{h_k}{2\pi} < \frac{1}{p}, \ k = \overline{0, r}, \tag{9}$$

which are sufficient for finding the general solution of the homogeneous Riemann problem (6) *in Hardy classes* $H_+^p \times {}_mH_-^p$*. For this case, the theory of Riemann problem has been well developed by I.I. Daniliuk [39]. So we obtain that if the inequalities* (8) *are true for some* $\alpha \in (0, 1)$*, then the general solution of homogeneous Riemann problem* (6) *in Hardy classes* $H_+^p \times {}_mH_-^p$ *has the form* $F(z) \equiv Z(z) P_m(z)$*, where* $Z(\cdot)$ *is a canonical solution, and* $P_m(\cdot)$ *is an arbitrary polynomial of degree* $k \leq m$.

On the contrary, if the inequalities (9) are true, then it is clear that there exists $\alpha \in (0, 1)$ such that the inequalities (8) are also true. Hence it follows that the assertion of Theorem 10 is valid.

Consider the non-homogeneous Riemann boundary value problem

$$F^{+}(\tau) - G(\tau) F^{-}(\tau) = f(\arg \tau), \tau \in \partial\omega, \tag{10}$$

in Morrey-Hardy classes $H_{+}^{p,\alpha} \times {}_{m}H_{-}^{p,\alpha}$, $\alpha \in (0, 1)$, $1 < p < +\infty$, where $f \in L^{p,\alpha}$ is some given function.

Let $Z(\cdot)$ be a canonical solution of a homogeneous problem corresponding to the problem (10). Consider the integral

$$F_1(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \left[Z^{+}\left(e^{it}\right)\right]^{-1} K_z(t) f(t)\, dt, \tag{11}$$

with Cauchy kernel $K_z(t) \equiv \frac{e^{it}}{e^{it}-z}$. The following theorem was proved in [38].

Theorem 11. *Let the coefficient $G(\cdot)$ of the problem (8) satisfy the conditions i), ii), and*

$$h_k = \theta(s_k + 0) - \theta(s_k - 0), \ k = \overline{1, r},$$

be the jumps of the function $\theta(t) \equiv \arg G\left(e^{it}\right)$ at the points of discontinuity

$$\{s_k\}_1^r \subset (-\pi, \pi); h_0 = \theta(-\pi) - \theta(\pi).$$

Assume that the following inequalities are fulfilled:

$$-\frac{\alpha}{q} \le \frac{h_k}{2\pi} < \frac{\alpha}{p}, k = \overline{0, r}. \tag{12}$$

Then the following assertions concerning the solvability of non-homogeneous problem (8) in the class $H_{+}^{p,\alpha} \times {}_{m}H_{-}^{p,\alpha}$ are true:

α) when $m \ge -1$, the problem (10) has a general solution $F(\cdot)$ of the form

$$F(z) = Z(z) P_m(z) + F_1(z),$$

where $Z(\cdot)$ is a canonical solution of the homogeneous problem (6), $P_m(\cdot)$ is an arbitrary polynomial of degree $k \le m$, $F_1(\cdot)$ is a particular solution of the form

$$F_1(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z^{+}(e^{it})} K_z(t)\, dt, \tag{13}$$

$K_z(\cdot)$ is a Cauchy kernel, and $f \in L^{p,\alpha}$ is an arbitrary function;

β) when $m < -1$, the problem (10) is solvable if and only if the orthogonality conditions

$$\int_{-\pi}^{\pi} \frac{f(t)}{Z^{+}(e^{it})} e^{ikt} dt = 0, k = \overline{1, -m-1}, \tag{14}$$

are true, and $F(z) \equiv F_1(z)$ *is a unique solution of this problem.*

This theorem has the following

Corollary 2. *Let all the conditions of Theorem 11 be fulfilled. Then the non-homogeneous problem* (10) *with arbitrary* $f \in L^{p,\alpha}$ *has a unique solution* $F_1(\cdot)$ *in the class* $H_+^{p,\alpha} \times {}_{-1}H_-^{p,\alpha}$*, which can be represented in the form of Cauchy type integral* (13).

Let's consider the case where the right-hand side of the problem (10) belongs to the space $\mathcal{M}^{p,\alpha}$. It follows directly from Theorem 9 that the boundary values $F_1^{\pm}(\cdot)$ of the function $F_1(z)$ defined by (11) also belong to $\mathcal{M}^{p,\alpha}$ if the inequalities (5) are true. Then the condition i) and Lemma 1 imply that the product $G(\cdot)\,F_1^-(\cdot)$ belongs to $\mathcal{M}^{p,\alpha}$. Consequently, similar to the proof of Theorem 11, we get the validity of the following theorem.

Theorem 12. *Let all the conditions of Theorem 11 be fulfilled. Then the following assertions concerning the solvability of the problem* (10) *with a right-hand side* $f(\cdot) \in \mathcal{M}^{p,\alpha}$ *in the class* $\mathcal{M}H_+^{p,\alpha} \times {}_m\mathcal{M}H_-^{p,\alpha}$ *are true:*

α) *when* $m \geq -1$*, the problem* (10) *has a general solution* $F(\cdot)$ *of the form*

$$F(z) = Z(z)\,P_m(z) + F_1(z)\,,$$

where $Z(\cdot)$ *is a canonical solution,* $P_m(\cdot)$ *is a polynomial of degree* $\leq m$*, and* $F_1(\cdot)$ *is a particular solution of the form* (13)*;*

β) *when* $m < -1$*, the problem* (10) *is solvable if and only if the orthogonality conditions* (14) *are true.*

Remark 2. *Again it should be noted that for* $\alpha \to 1-0$ *the inequalities* (12) *become*

$$-\frac{1}{q} \leq \frac{h_k}{2\pi} < \frac{1}{p}, k = \overline{0,r}. \tag{15}$$

The inequalities (14) are sufficient for finding the general solution of non-homogeneous problem (10) in classical Hardy classes $H_+^p \times {}_mH_-^p$. The theory of this problem has been developed by I.I. Daniliuk [39]. So, if the inequalities (12) hold, then the assertions α) and β) of Theorem 11 are true with regard to the solvability of non-homogeneous problem (10) in the class $H_+^p \times {}_mH_-^p$. On the contrary, if the inequalities (12) hold, then there exists $\alpha \in (0,1)$ such that the inequalities (14) are true, and hence the assertions of Theorem 11 are valid.

4. Basicity of perturbed system of exponents in Morrey-Lebesgue space

Consider the following dual system of exponents:

$$\left\{A(t)\, e^{i\,nt}; B(t)\, e^{-i\,nt}\right\}_{n\in Z_+,k\in N}, \tag{16}$$

with complex coefficients

$$A(t) = |A(t)|\, e^{i\alpha(t)}; B(t) = |B(t)|\, e^{i\beta(t)},$$

on an interval $[-\pi,\pi]$, where $Z_+ = \{0\}\cup N$. We require the fulfillment of the following conditions

$\alpha)$ $A^{\pm}$; $B^{\pm} \in L_\infty \equiv L_\infty(-\pi,\pi)$;

$\beta)$ $\theta(t) \equiv \beta(t) - \alpha(t)$ *is a piecewise continuous function on* $[-\pi,\pi]$ *with the points of discontinuity* $\{s_k\}_1^r : -\pi < s_1 < ... < s_r < \pi$, *and* $h_k = \theta(s_k+0) - \theta(s_k-0)$, $k=\overline{1,r}$ *- are the jumps of this function at the points* s_k, *and let* $h_0 = \theta(-\pi) - \theta(\pi)$.

Let

$$G\left(e^{it}\right) = \frac{A(t)}{B(t)}, t\in[-\pi,\pi].$$

Take $\forall f \in \mathcal{M}^{p,\alpha}$ and consider the following Riemann boundary value problem in classes $\mathcal{M}H_+^{p,\alpha} \times {}_{-1}\mathcal{M}H_-^{p,\alpha}$:

$$F^+(\tau) - G(\tau)\, F^-(\tau) = A^{-1}(\arg\tau)\, f(\arg\tau), \tau\in\gamma. \tag{17}$$

Assume that the following inequalities are fulfilled:

$$-\frac{\alpha}{q} \le \frac{h_k}{2\pi} < \frac{\alpha}{p}, \quad k=\overline{0,r}. \tag{18}$$

We apply Theorem 12 to the solution of the problem (17). From the condition $\alpha)$ and Lemma 1 it follows that the function $A^{-1}(\cdot)\, f(\cdot)$ belongs to the space $\bar{L}^{p,\alpha}$. Therefore, if the inequality (18) is fulfilled, then, as it follows from Theorem 12, the problem (17) is uniquely solvable in the class $\mathcal{M}H_+^{p,\alpha} \times_{-1} \mathcal{M}H_-^{p,\alpha}$, and this solution can be represented as an integral

$$F(z) = \frac{Z(z)}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{A(t)\, Z^+(e^{it})}\frac{dt}{1-ze^{-it}}.$$

It is absolutely clear that the following inclusion is true

$$F^+(\cdot) \in \mathcal{M}_+^{p,\alpha}; F^-(\cdot) \in {}_{-1}\mathcal{M}_-^{p,\alpha}.$$

By Theorem 7, the systems $\left\{e^{i\,nt}\right\}_{n\in Z_+}$, $\left\{e^{-i\,nt}\right\}_{n\in N}$ form a basis for the spaces $\mathcal{M}_+^{p,\alpha}$ and ${}_{-1}\mathcal{M}_-^{p,\alpha}$, respectively. Let us expand the functions $F^+(\cdot)$ and $F^-(\cdot)$ on these systems. We have

$$F^+\left(e^{it}\right)=\sum_{n=0}^{\infty}a_n e^{i\,nt};F^-\left(e^{it}\right)=\sum_{n=1}^{\infty}b_n e^{i\,nt}.$$

Taking into account these decompositions in (17), we obtain that the function $f(\cdot)$ has an expansion on the system (16) in $\mathcal{M}^{p,\alpha}$:

$$f(t)=A(t)\sum_{n=0}^{\infty}a_n^+ e^{i\,nt}+B(t)\sum_{n=1}^{\infty}a_n^- e^{-i\,nt}, \tag{19}$$

where $a_n^+=a_n$, $a_n^-=-b_n$. Let us find an expression for the coefficients $\{a_n^\pm\}$. Let

$$Z(z)=\sum_{n=0}^{\infty}c_n^+ z^n,\ \ |z|<1,$$

be a Taylor expansion of the function $Z(\cdot)$ in the neighborhood of the point $z=0$. We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{A(t)\,Z^+\left(e^{it}\right)}\frac{dt}{1-ze^{-it}}=\sum_{n=0}^{\infty}d_n^+ z^n,$$

where

$$d_n^+=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-i\,nt}}{A(t)\,Z^+\left(e^{it}\right)}f(t)\,dt,\ n\in Z_+.$$

As a result, for $F(\cdot)$ we have the decomposition

$$F(z)=\sum_{n=0}^{\infty}A_n^+ z^n,\ \ |z|<1,$$

where $A_n^+=\sum_{k=0}^{n}c_{n-k}^+ d_k^+$. Taking into account the expression for d_n^+, we have

$$A_n^+=\int_{-\pi}^{\pi}\overline{v_n^+(t)}f(t)\,dt,$$

where

$$v_n^+(t)=\frac{1}{2\pi}\sum_{k=0}^{n}\bar{c}_{n-k}^+\frac{e^{+i\,kt}}{\overline{A(t)Z^+\left(e^{it}\right)}},\ n\in Z_+.$$

Similarly, let us expand the function $F(\cdot)$ in a Taylor series in the neighborhood of $z=\infty$. Again, using the expression for $F(\cdot)$, we have

$$Z(z)=\sum_{n=0}^{\infty} c_n^- z^{-n},\quad |z|>1.$$

Also we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{A(t)\,Z^+(e^{it})}\frac{dt}{1-ze^{-it}}=\sum_{n=1}^{\infty} d_n^- z^{-n},$$

where

$$d_n^-=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\,nt}}{A(t)\,Z^+(e^{it})}f(t)\,dt,\quad n\in N.$$

So, $F(\cdot)$ has an expansion

$$F(z)=\sum_{n=1}^{\infty} A_n^- z^{-n},\quad |z|>1,$$

where

$$A_n^-=\sum_{k=1}^{n} c_{n-k}^- d_k^-.$$

Similar to the previous case, for A_n^- we have $A_n^-=\int_{-\pi}^{\pi}\overline{v_n^-(t)}f(t)\,dt$, where

$$v_n^-(t)=-\frac{1}{2\pi}\sum_{k=1}^{n}\bar{c}_{n-k}^-\frac{e^{-i\,kt}}{\overline{A(t)Z^+(e^{it})}},\quad n\in N.$$

On the other hand, it is easy to see that the relations

$$a_n^+=\frac{1}{2\pi}\int_{-\pi}^{\pi}F^+\left(e^{it}\right)e^{-i\,nt}dt,$$

$$A_n^+=\frac{1}{2\pi r^n}\int_{-\pi}^{\pi}F\left(re^{it}\right)e^{-i\,nt}dt,\ 0<r<1,\ \forall n\in Z_+.$$

are true. Then from Theorem 5 it directly follows that $A_n^+=a_n^+$, $\forall n\in Z_+$. Similarly we obtain $A_n^-=a_n^-$, $\forall n\in N$.

And now, as $f(t)$ we take $f(t)\equiv A(t)\,e^{in_0t}$, where $n_0\in Z_+$ is an arbitrary fixed number. In this case the solution of the problem (17) is the following function:

$$\Phi^{+}(z)=\begin{cases} z^{n_0}, & |z|<1 \\ 0, & |z|>1. \end{cases}$$

As it follows from Theorem 12, the problem (17) is uniquely solvable in classes $\mathcal{M}H_{+}^{p,\alpha}\times {}_1\mathcal{M}H_{-}^{p,\alpha}$. Therefore, comparing functions $F(\cdot)$ and $\Phi^{+}(\cdot)$, we have

$$A_n^{+}=\begin{cases} 1, & n=n_0, \\ 0, & n\neq n_0, \end{cases} \quad A_n^{-}=0, \forall n\in N,$$

i.e.

$$\int_{-\pi}^{\pi} A(t)\, e^{in_0t}\bar{v}_n(t)\, dt=\delta_{nn_0}, \forall n,n_0\in Z_{+},$$

$$\int_{-\pi}^{\pi} A(t)\, e^{in_0t}\bar{v}_n^{-}(t)\, dt=0, \forall n_0\in Z_{+}, \forall n\in N.$$

Similarly, taking $f(t)=B(t)\,e^{-in_0t}$, we obtain that in this case the solution of the problem (17) is the following function

$$\Phi^{-}(z)=\begin{cases} 0, & |z|<1, \\ z^{-n_0}, & |z|>1. \end{cases}$$

Again, by similar considerations we obtain

$$\int_{-\pi}^{\pi} B(t)\, e^{-in_0t}\bar{v}_n^{+}(t)\, dt=0, \forall n_0\in N, \forall n\in Z_{+};$$

$$\int_{-\pi}^{\pi} B(t)\, e^{-in_0t}\bar{v}_n^{-}(t)\, dt=\delta_{nn_0}, \forall n,n_0\in N.$$

From these relations it follows that if the inclusion $\{v_n^{+};v_n^{-}\}_{n\in Z_+,k\in N}\subset(\mathcal{M}^{p,\alpha})^{*}$ is true, then the system (16) is minimal in $\mathcal{M}^{p,\alpha}$. It suffices to show that

$$v_n^{+};v_k^{-}\in L^{q,\alpha}, \forall n\in Z_{+}, \forall k\in N.$$

We have

$$\left|Z^{+}\left(e^{it}\right)\right|^{-1}\sim |t^r-\pi^r|^{\frac{h_0}{2\pi}}\prod_{k=1}^{r}|t-s_k|^{\frac{h_k}{2\pi}}, t\in[-\pi,\pi].$$

Let the inequalities

$$\frac{h_k}{2\pi} \geq \frac{-\alpha}{q}, k = \overline{0, r}.$$

be fulfilled. Then by Lemma 4 it follows that $\left|Z^+\left(e^{it}\right)\right|^{-1}$ belongs to $L^{q,\alpha}$. Hence, from the expressions for $v_n^{\pm}$ it follows the inclusion $\{v_n^{\pm}\} \subset L^{q,\alpha}$. As a result, we obtain that the system (16) is minimal in $\mathcal{M}^{p,\alpha}$. It is clear that the function $f(\cdot)$ has a unique expansion of the form (19) on this system. Thus, the following theorem is true.

Theorem 13. *Let the functions $A(\cdot)$ and $B(\cdot)$ satisfy the conditions α) and β). If the inequalities*

$$-\frac{\alpha}{q} \leq \frac{h_k}{2\pi} < \frac{\alpha}{p}, k = \overline{0, r},$$

are fulfilled, then the system (16) *forms a basis for* $\mathcal{M}^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$.

Consider the particular cases of this theorem. Let $\beta \in R$ be some parameter and consider the following system of exponents

$$\left\{e^{i(n+\beta \, sign \, n)t}\right\}_{n \in Z}. \tag{20}$$

In this case, we have $A(t) = e^{i\beta t}$; $B(t) = e^{-i\beta t}$. Thus, the function $\theta(t) = -2\beta t$ doesn't have any point of discontinuity on $(-\pi, \pi)$ and $h_0 = \theta(-\pi) - \theta(\pi) = 4\beta\pi$. Applying Theorem 13 we obtain

Corollary 3. *Let the inequality*

$$-\frac{\alpha}{2q} \leq \beta < \frac{\alpha}{2p},$$

be fulfilled. Then the system of exponents (20) *forms a basis for* $\mathcal{M}^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$.

Let us consider the other particular case

$$\left\{e^{i(n+\beta \, sign \, n \, sign \, t)t}\right\}_{n \in Z}, \tag{21}$$

where $\beta \in R$ is some parameter. We have $A(t) = e^{i\beta|t|}$; $B(t) = e^{-i\beta|t|}$. In this case the function $\theta(t) = -2\beta|t|$ doesn't have any point of discontinuity, $h_0 = \theta(-\pi) - \theta(\pi) = 0$, and all the conditions of Theorem 13 are fulfilled.

Corollary 4. *Let β be an arbitrary real parameter. Then the system of exponents* (21) *forms a basis for* $\mathcal{M}^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$.

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