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ON THE CLASS OF LINEAR OPERATORS ASSOCIATED WITH THE MULTIPARAMETER SPECTRAL THEORY

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Multiparameter Spectral (MPS) theory essentially consists of a new multidimensional analogue of classical spectral theory of operators-the multidimensionality is associated with the presence of many (two or more) spectral parameters. MPS theory covers a broad class of problems associated with the operator families given in general form, as well as operators represented by differential and other types of functional equations. MPS theory has been stimulated by mathematical physics, particularly by abstracting from the method of separation of variable to solve boundary value problems; many of higher trancendent functions appear on this track [[1]-[4]].

The present work is devoted to the study and discussion of a class of operators originated from MPS theory, some historical notes on interpretations and misrepresentations of this class of operators and MPS problems in general.

Let me introduce MPS system of linear operators depending on several spectral parameters:

$$P(\lambda) = \Big(P_1(\lambda), ..., P_n(\lambda)\Big)$$

Where

$$P_i(\lambda) = A_j - \lambda_1 B_{j1} - ... - \lambda_n B_{jn},\ j = 1, 2, ..., n.$$

The operators A_j, B_{jk} act in complex Hilbert spaces H_j, $j = 1, 2, ..., n$ and $\lambda = (\lambda_1, ..., \lambda_n)$ is a multidimensional spectral parameter.

Let me consider the system of equations

$$P_j(\lambda)x_j = 0,\ j = 1, 2, ..., n \tag{1}$$

A point $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ is called an eigenvalue of the family $P(\cdot)$ or an eigenvalue of the multiparameter spectral problem, if there exist nonzero elements $x_1, ..., x_n$ of all equations (1), decomposable tenzor $x = x_1 \otimes ... \otimes x_n$ is called the eigenvector of MPS problem (or family $P(\cdot)$), associated with the eigenvalue λ.

For the sake of simplicity, I am focusing on the Hilbert spaces case without discussing MPS theory in general Banach or wider (more general) spaces, unless the opposite is necessary or required.

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Let me start with a brief history of MPS theory. Attention to the study of basic concepts of MPS theory, including the use of operators acting in tensor space $H = H_1 \otimes ... \otimes H_n$ of initial spaces H_j, $j = 1, 2, ..., n$ was drawn by the work of F.V. Atkinson in 1968 [5]. Before that, two-parameter (occasionaly three- and n-parameter) problems were discussed for various second order differential operators in the second half of the 19th century and in the first quarter of the

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20th century by F. Klein, M.Boxer, J.Yoshikawa, R, Richardson, A.C. Dixon, E.Hilb, D.Hilbert, J. Anna Pell, and R.D. Carmichael [[5]-[7]].

A valuable study of spectral theory of problem

$$(C_1 + C_2)u = \lambda(D_1 + D_2)u$$

where $C_1(C_2)$ and $D_1(D_2)$ depend on variable $x(y)$
was done by H.O. Cordes in the mid-1950s [8]. He applied the theory of tensor product of Hilbert spaces and linear operators to this equation with the aim of separating variables. Through the process of substantiation of the method of separation of variables, he came upon a special class of selfadjoint two-parameter spectral problems. In terms of MPS theory he supposed (within other requirements) fulfillment of definiteness condition and proved the selfadjointness and commutativeness of the two, so-called, separating operators in the sence of commutativeness of their resolvents. His principal achivement was working out the construction method of resolutions of the identity of these two separating selfadjoint operators.

F.Atkinson in his monograph [9] studied, first of all, finite-dimensional MPS problem and its link to the family of operators

$$\Gamma_j = \Delta_0^{-1}\Delta_j, \ j = 1, 2, ..., n,$$

where $\Delta_0, \Delta_1, ..., \Delta_n$ are determinants, corresponding to Cramer determinants for the system of linear algebraic equations compiled from operator coefficients of the system (1), with tensor product as an operation of product. All operators $\Gamma_1, ..., \Gamma_n$ act in the same space-tensor space $H_1 \otimes ... \otimes H_n$. Atkinson proved a theorem of expansion in finite dimensional case and then he extended expansion theorem by the method of approximation for an infinite dimensional case with a discrete spectrum. Expansion theorems play a central role in all spectral theories, including the multiparameter spectral expansion problems (both general and differential operators cases), for main results and development here see [[10]-[13]]

In the 1970s, interest in MPS theory essentially increased. Group of mathematicians from Azerbaijan joined this study (mainly the author of this work and his students) and the network of experts from USA, Canada, UK, Germany and Azerbaijan (one of the republics of the Soviet Union then) worked out the expansion and deepening of the theory. The Azerbaijani group developed MPS theory in several directions, particularly expansion theorems for eigenvectors of singular multiparameter differential operators with arbitrary order, general theory of separating system of operators linked to MPS problem, the joint Taylor spectrum of the separating system of operators associated with multiparameter system, construction of spectral measures of separating system for the selfadjoint case, oscillation theorems for various MPS problems, theory of multiparameter deficiency indices for general and differential operators, etc. [[10]-[11], [14]-[16]].

Unfortunately, beginning from the mid-1980s, misinterpretations can be found in the history of the development of MPS theory, often accompanied by untruthful statements. Some dissonance with inconsistent arguments arouse on the corner of the scene, contradicting the friendly academic atmosphere within the international network. I will discuss some of them - one is them is an article written by Jalal Allakhverdiyev [17].

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Let us remember once again the family of determinant operators $\Delta_0, \Delta_1, ..., \Delta_n$ acting on the tensor product of initial spaces by the Cramer rule where the tensor product of the operator-components, i.e. corresponding operators $A_j, \ B_{j,k}$ is used instead of ordinary multiplication. For instance, by definition

$$\Delta_0 = \sum_{\sigma} \varepsilon_{\sigma} B_{1\sigma(1)} \otimes ... \otimes B_{n\sigma(n)}$$

where $\sigma = (\sigma(1), ..., \sigma(n))$ runs through all permutations of $\{1, 2, ..., n\}$ and ε_σ is the signature of σ.

By definition, the family of operators

$$T_j(\lambda) = \Delta_j - \lambda_j \Delta_0, \; j = 1, 2, ..., n$$

form the separating system of MPS problem (for the family $P(\cdot)$). In this way, the separation of the spectral parameters can be proceeded, i.e. an equivalence can be established between the original MPS problem (for multiparameter family $P(\cdot)$) and spectral problem for the family of operators $T_1(\lambda_1), ..., T_n(\lambda_n)$ each of which depends only on their own corresponding single parameter. For the "regularity" or "naturalness" one must suppose that operator Δ_0 (denominator of Cramer formula) is invertible. Then the operators

$$\Gamma_j = \Delta_0^{-1} \Delta_1, \; j = 1, 2, ..., n$$

form the regular separating system for the multiparameter family $P(\cdot)$. In the case of Hilbert spaces, strong regularity comes with the uniform definiteness condition, i.e. operator Δ_0 is a uniform positive operator in the tensor space H:

$$\Delta_0 \geq \alpha I, \; \alpha > 0.$$

Thus, the commutative family of operators $\Gamma_1, ..., \Gamma_n$ acting in the tensor product $H = H_1 \otimes ... \otimes H_n$ helps attempt to separate spectral parameters $\lambda_1, ...\lambda_n$ through an equivalence between the joint spectrum of the family and original MPS problem. The essential chapters of MPS theory and related problems are based on the study of this fruitful interconnection.

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MPS theorem on expansion is not extended for "normal" (or "unitary") operators, even no mention of what normal MP system is. At glimpse, it seems that if $A_1, ..., A_n$ are bounded normal operators and $B_{j1}, ..., B_{jn}, \; j = 1, 2, ..., n$ are selfadjoint operators with uniform definiteness condition, then the separating system of MPS problem $\Gamma_1, ..., \Gamma_n$ may become normal family of operators (family of normal operators). However, it is wrong even in the two-dimensional case [[10], p. 55].

In non-selfadjoint case of operators $A_1, ..., A_n$ (all or some of them), for the study of adjoint separating system of the family $\Gamma_1, ..., \Gamma_n$ and its extension I have introduced a class of linear operators (in general, unbounded) as the strong limit of certain sequences of bounded operators [[7], [18]].

Definition 1 ([7], [18]). . *Linear operator A acting in Hilbert space H with dense domain $D(A)$ and image $R(A)$ is said to be strongly genetic, if there exists a sequence (J_m) of bounded operators on H such that:*

- **(a):** *$J_m A$ is a bounded operator with domain $D(A)$, $m = 1, 2, ...$*
- **(b):** *$s - \lim_{m\to\infty} J_m x = x, \; x \in R(A)$ if, moreover*
- **(c):** *$s - \lim_{m\to\infty} J_m^* x = x, \; x \in H,$*

Then A is said to be strongly bigenetic operator.

It is clear that every normal (particularly, selfadjoint) operator is strongly bigenetic. Moreover, it is easy to see that arbitrary closable operator is strongly bigenetic. Jalal Allakhverdiyev in his article [17] asserts that the opposite is true as well, i.e. strongly bigenetic operator is closable. However, his claim is not true. He brings in a general statement

Theorem A[1]**.** *Let F_1 and F_2 be Fréchet spaces and for a linear operator $A : F_1 \to F_2$ and there exists sequence of bounded operators A_n such that $A_n f \to Af$ weakly for $f \in D(A)$. Then A is closable.*

Here is the simple **Counterexample** to this claim.

Let be $F_1 = F_2 = L^2[0, 1]$ and $D(A) = C[0, 1]$

Then let me introduce $(Af)(t) = f(0)1(t)$, where $1(t)$ is a constant function from $L^2[0,1]$ equal to one. This operator A is non closable. Indeed, the sequence $f_n(t) = (1-t)^n$ from $D(A)$ tends to zero in $L^2[0,1]$, but $(Af_n)(t) = 1(t)$.

Let me consider the sequence of operators

$$(A_n f)(t) = \Big(n\int_0^{1/n} f(t)dt\Big)1(t).$$

It is obvious that operators A_n are bounded and $A_n f \to Af$ strongly for $f \in D(A)$.

Thus, this Theorem $\mathbf{A}^1$ (of Jalal Allakhverdiyev) is wrong. The above example (counterexample to the Theorem $\mathbf{A}^1$) demonstrates that it is impossible to repair it - even introducing this theorem in Hilbert spaces and even requiring the strong convergence of $A_n f$ to Af.

Let me go further and formulate another statement using the same article [17] by Jalal Allakhverdiyev.

Theorem $\mathbf{A}^2$. *Let be A linear operator from F_1 to F_2 and there exists sequences of linear operators E_n such that E_nA are bounded and $E_ng \to g$ weakly for $g \in R(A)$. Then operator A is closable.*

Erroneousness of this declaration follows from the next theorem (again, neither Hilbert space nor strong convergence helps to save this hopeless statement)

Theorem 1. *There exists non - closeable operator A acting in Hilbert space H with dense domain $D(A)$ such that, there exists sequence of bounded operators J_m, $m = 1, 2, ...$ possessing relevant properties a) and b).*

Proof. Let me consider l_2 of all sequences of complex numbers $\{x_k\}$ such that $\sum |x_k|^2 < \infty$ and let be $D(A) = l^1 = \Big\{\{x_k\} : \sum\limits_{k=1}^{\infty} |x_k| < \infty\Big\}$

Suppose that A is an operator $Ax = y$, where

$$x = \{x_k\}_{k=1}^{\infty},\ y = \{y_k\}_{k=1}^{\infty},\ y_k = \frac{1}{k}\sum_{s=k}^{\infty} x_s$$

I allege that operator A is non-closeable.

Let $x^{(p)}$ be the sequences from elements of $D(A)$ defined with the formula

$$x^{(p)} = \Big(\frac{1}{p}, \frac{1}{p+1}, ..., \frac{1}{p+\tilde{p}}, \frac{1}{(p+\tilde{p}+1)^2}, \frac{1}{(p+\tilde{p}+2)^2}, ...\Big)$$

Where $\tilde{p}$ is minimal natural number with property

$$\frac{1}{p} + \frac{1}{p+1} + ... + \frac{1}{p+\tilde{p}} \geq 1$$

It is easy to see, that $x^{(p)} \to 0$ and

$$Ax^{(p)} \to \Big(1, \frac{1}{2}, ..., \frac{1}{k}, ...\Big)$$

when $p \to \infty$.

Indeed, if $f_n \to f$ weakly and $\|f_n\| \to \|f\|$, then $f_n \to f$ strongly (incidentally, in order to prove that A is non-closable, it is sufficient to prove the weakly convergence both of $x^{(p)}$ and $Ax^{(p)}$ to specified limits due to weak closeness of each strong closed subspace of Hilbert space).

Let me now introduce operators I_m by formula

$$J_m x = \Big(x_k - \frac{m+k-1}{k}\cdot x_{m+k-1}\Big)_{k=1}^{\infty}$$

It is clear, that these operators are bounded ($\|I_m\| \leq m+1$) and

$$J_m Ax = \Big(\frac{1}{k}\sum_{s=k}^{m+k-2}\Big)_{k=1}^{\infty},\ x \in D(A)$$

Therefore $\|J_m Ax\| \leq (m-1)\|x\|$, and in addition $J_m Ax \to Ax$ strongly for all $x \in D(A)$. This completes the proof. □

Let me verify that constructed in this way (as in the previous theorem) operator A satisfies also the following property:

Theorem 2. *Let $Q_m,\ m = 1, 2, ...$ be bounded operators satisfying properties $a)$ and $b)$ from the definition of strongly genetic operator (replacing J_m with Q_m) and J_m be an operator which constructed in Theorem 1. Then there exists an element x_0 such that*

$$\lim_{m\to\infty} \|Q_m^* x_0 - x_0\| \neq 0$$

Note In terms of the article [18] the operator A is genetic, but not bigenetic. In particular case, **it follows from this theorem that the rest of Jalal Allakhverdiyev's inferences on class of operators introduced in** [18] **is wide of the mark as well.**

Proof. Starting with the proof of Theorem 2, it is easy to see that the image $R(A)$ of the constructed operator A is dense in H; indeed,

$$e_1 = A(1,0,0,...),\ e_2 = A(-2,2,0,0,...),\ ...,\ e_n = A(0,...,0,-n,n,0,0,...),$$

for the standard orthonormal basis $e_n = (0, ..., 0, 1, 0, ...)$ with 1 in a place of number n. Then, considering the property $b)$ one can conclude that

$$\lim_{m\to\infty}([Q_m - J_m]x, y) = 0 \ for\ all\ x, y \in H,$$

and therefore $(Q_m^* - J_m^*)y \to 0$ weakly with $m \to \infty$.

Now if one suppose contradiction, i. e. $\|Q_m^* x - x\| \to 0$ when $m \to \infty$ for all $x \in H$, it will take to

$$\lim_{m\to\infty} \Big(y - J_m^* y, y\Big) = 0.$$

However, the vector $y_0 = \Big(1, \frac{1}{2}, ..., \frac{1}{k}, ...\Big)$ does not satisfy this limit-relation. Indeed, it is easy to see that

$$J_m^* = \Big(x_1, ..., x_{m-1}, x_m - mx_1, x_{m+1} - \frac{m+1}{2}x_2, ..., x_{m+k-1} - \frac{m+k-1}{k}x_k, ...\Big)$$

And therefore

$$y_0 - J_m^* y_0 = \Big(0, 0, ..., 0, m, \frac{m+1}{4}, ...\Big)$$

or

$$\Big(y_0 - J_m^* y_0, y_0\Big) = 1 + ...$$

This proves the theorem. □

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Note: I personally sent all these Counterexamples, Examples and Theorems to Jalal Allakhverdiyev in preprint form, obviously demonstrating the inaccuracy of his claims and statements. I supposed he would send a letter of apology to the journal, and in that case, there would be no need to produce this article to disprove the results of Jalal Allakhverdiyev's discussed article. Unfortunately, he moved in the opposite direction - he and his student M.B. Rahimov activated their attack strategies; they sent strange letters to various people - concerned or not. Then M.B. Rahimov published an article in support of Jalal Allakhverdiyev (as he indicated there) in a non peer-reviewed marginal and trifling collection (Linear Operators and their Applications, collection of articles, Azerbaijan State University, Baku, 1986), with the intention of maligning the author of this article. Unfortunately, all these were beyond academic ethics. The article itself contained a complete misunderstanding of basic definitions, general ideas, history of development and achievements of multiparameter spectral theory as well as ill-grounded claims and false accusations, intentional distortions and misrepresentations of facts. Some comments about it were included in the above-mentioned preprint work [19]. An interesting paper [20], written by a well-known investigative journalist from the central newspaper office of those times, was devoted to this unpleasant event with the background information and motives of these actions.

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Abstract

Multiparameter Spectral Theory covers a broad class of problems linked to operator families depending on many spectral parameters given in general forms, as well as operators represented by differential and other types of functional equations. This theory has been stimulated by mathematical physics, particularly by abstracting from the method of separation of variables to solve boundary value problems. The present work is devoted to the study and discussion of a class of operators originated from Multiparameter Spectral Theory, some historical notes on the interpretations and misrepresentations of this class of operators and multiparameter problems in general.

Key words: spectral theory of operators, multiparameter spectral theory, tensor production, genetic operators, closable operators, Hilbert space.